Mathematics 126

Final Exam

Dec. 8, 2003

• Calculators are allowed *only* for numerical calculations.

• There are three sheets of scratch paper attached at the end of the exam. Use them and but do not tear them off the exam in doing so, and hand them in together with the exam.

• Show your work; clearly write down each step in your calculations/reasonings.

1. Evaluate the following definite and indifinite integrals.

a)

$$\int_1^e \frac{(\ln x)^2}{x}\ dx$$

b)

$$\int_2^3 \frac{1}{x^2-1}\ dx$$

c)

$$\int_0^1 \sqrt{5x+4}\ dx$$

d)

$$\int x\ e^x\ dx$$

e)

$$\int_2^6 \sqrt{x-2}\ dx$$

Note: this is an improper integral.

f)

$$\int_0^1 \frac{x^3}{x^2+1}\ dx$$

2. There is an *upside-down* pyramid-shaped container whose square base has its side-length 20 in. and its height 10 in.. Let h be the height measured from the bottom of the container, that is the tip of the pyramid is at $\{x = 0\}$

a) Write down the area $A(x)$ of the cross-section of the container at height x.

b) Write down the volume $V(h)$ of water of height h in the contianer as an integral of $A(x)$, and evaluate the integral.

c) Find the height of water when the container is $\frac{1}{8}$ filled, (when the volume of the water is $\frac{1}{8}$ of the capacity of the container.)

d) Find the height h_0 so that $V'(h_0) = 64$. (Hint: apply the fundamental theorem of calculus to $V(h)$.)

3. Sketch the region enclosed by the curves $y = |x|$, $y = x^2 - 2$. Find the area of the region.

4. Let $P = (1, 0, -1)$ and $Q = (2, 1, 2)$.
a) Take the dot product of the two vectors $\vec{OP}$ and $\vec{OQ}$. What can you say about the angle between the two vectors?

b) Take the cross product of the two vectors $\vec{OP}$ and $\vec{OQ}$.

c)Find an equation of the plane containing O, P and Q.

5. Determine whether each *series* converges or not. Write down your reasoning (what test you used etc..)

a)

$$\sum_{n=1}^{\infty} a_n, \text{ where } a_1 = 1 \text{ and } a_{n+1} = \frac{2n^2}{3n^2 + n} a_n.$$

b)

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1 + 2\sqrt{n}}$$

6. **a)** Using the integral of the function $1/x$, show that the n-th partial sum s_n of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ satisfy the following inequality

$$s_n \leq 1 + \ln n.$$

b) Even though the harmonic series diverges, it does so *very slowly.* Use part **a)** to show that the sum of the first million (10^6) terms is less than 15. (you may use the fact that $\ln 10 = 2.32\ldots$.)

7. a) Find the Taylor series for the function e^x centered at 1.

b) Find an approximate value of $e^{0.9}$ by using the second order Taylor's polynomial T_2 of e^x centered at $a = 1$.

c) Find an upper bound for the error $R_2(0.9) = e^{0.9} - T_2(0.9)$ for the approximation of the part **a**. Hint: You can either use Alternating Series Error Estimate or Taylor's Theorem.

8. a) Find the Taylor series of $\sin x$ and $\cos x$ at $a = 0$. (First write down the first four or five terms, and recognize the pattern to come up with the general formuli.)

b) Differentiating the power series for $\sin x$ from **a)** and check that it equals to the power series for $\cos x$ as you expect (recall $(\sin x)' = \cos x$.)