The University of Alabama System

Joint Ph.D Program in Applied Mathematics

Linear Algebra and Numerical Linear Algebra JP Exam

August 2024

Instructions:

- This is a closed book examination. Once the exam begins, you have three and one half hours to do your best. You are required to do **seven of the eight problems for full credit.**
- Each problem is worth 10 points; parts of problems have equal value unless otherwise specified.
- Justify your solutions: cite theorems that you use, provide counter examples for disproof, give explanations, and show calculations for numerical problems.
- Begin each solution on a new page and write the last four digits of your university **student ID number**, and problem number, on every page. Please write only on one side of each sheet of paper.
- The use of calculators or other electronic gadgets is not permitted during the exam.
- Write legibly using dark pencil or pen.

1. Let A be an isometry on a finite-dimensional real inner product space V which satisfies $A^2 = -I$. Prove that for every vector v in V, Av is orthogonal to v.

2. Let A be an $n \times n$ positive semi-definite matrix.

   (a) Show that

   $$\|(I - A)(I + A)^{-1}\mathbf{x}\|_2 \leq \|\mathbf{x}\|_2, \mathbf{x} \in \mathbb{C}^n.$$

   (b) Show that $\mathbf{x} \in \text{null}A$ is equivalent to

   $$(I - A)(I + A)^{-1}\mathbf{x} = \mathbf{x}$$

3. Let A and B be two complex square matrices, and suppose that A and B have the same eigenvectors. Show that if the minimal polynomial of A is $(x+1)^2$ and the characteristic polynomial of B is x^5, then $B^3 = 0$.

4. Consider the vector space $\mathcal{M}_{n\times n}(\mathbb{R})$ of $n \times n$ real matrices. Consider the linear map $T : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathcal{M}_{n\times n}(\mathbb{R})$ given by $T(A) = A^T$ for all $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Here A^T denotes the transpose of A.

   (a) Find the characteristic polynomial and minimal polynomial of T.

   (b) Find the Jordan form of T.

5. Consider the linear system

   $$Ax = b$$

   where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, m \leq n$, and $\text{rank}(A) = m$. Let $\Omega = \{x \in \mathbb{R}^n | Ax = b\}$. Prove the following theorems.

   (a) $\text{Range}(A^T)^\perp = \text{Ker}(A)$

   (b) Let $x, \hat{x} \in \Omega$. If $\hat{x}$ is the orthogonal projection of x onto $\text{Range}(A^T)$, then $\hat{x} = A^T(AA^T)^{-1}b$.

   (c) The unique solution to $Ax = b$ that minimizes the norm $\|x\|$ is $\hat{x} = A^T(AA^T)^{-1}b$, i.e.

   $$\hat{x} = \underset{x\in\Omega}{\text{argmin}} \quad \|x\|.$$

(d) Moreover, show that $x - \hat{x} = P_2 x$ where $P_2 = I - P_1$ (i.e., P_2 is the complementary projector of P_1), P_1 is a projection onto Range(A^T) along Ker(A) and P_2 is a projection onto Ker(A) along Range(A^T).

6. Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix and let the vectors b and Δb be such that

$$\begin{aligned} Ax &= b \\ A(x+\Delta x) &= b + \Delta b\,. \end{aligned}$$

(a) Show that

$$\frac{\|\Delta x\|_2}{\|x\|_2} \le \kappa_2(A)\left(\frac{\|\Delta b\|_2}{\|b\|_2}\right)$$

where $\kappa_2(A) := \|A\|_2\|A^{-1}\|_2$, the condition number of A.

(b) Give an explanation of the bound in (a) with respect to the significant role of the condition number in the senstivity analysis of the linear systems. Hint: use SVD of diagonal matrix A as an example.

7. (a) Give an explanation of what is meant by the least squares solution of $Ax = b$, where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$ and $m > n$.

(b) Find the least squares solution of the system

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}.$$

(c) Also compute the norm of the minimal residual vector.

8. Let $A \in \mathbb{R}^{n\times n}$ be given, symmetric, and assume that the eigenvalues of A satisfy

$$|\lambda_1| > |\lambda_2| \ge \ldots \ge |\lambda_{n-1}| \ge |\lambda_n|.$$

Let $z \in \mathbb{R}^n$ be given. Under what conditions on z does the following hold, theoretically? (Be sure to actually show that it holds!)

$$\lim_{k\to\infty} \frac{z^T A^{k+1} z}{z^T A^k z} = \lambda_1$$

Under what conditions on z does this hold, *as a practical matter?* Explain fully for full credit.