Joint Program Exam in Mathematical Analysis
August 12, 2024

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet.Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 7 problems. All the problems are weighted equally.You need to do ALL of them for full credit.

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems.You do not need to reprove the theorems you used.

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, so that $f(x) \geq 0$. Prove that if

$$\int_a^b f(x)dx = 0,$$

then $f(x) = 0$ for all $x \in [a, b]$.

2. Let $f : (-1, 1) \to \mathbb{R}$ be a C^2 function, with $f''(0) \neq 0$. By the mean value theorem, we know that, for each $x \in (-1, 1)$, there is $\theta = \theta(x)$, so that

$$f'(\theta) = \frac{f(x) - f(0)}{x}.$$

Prove that the limit

$$\lim_{x \to 0} \frac{\theta(x)}{x}$$

exists and compute it.

3. Prove that $f(x) = \sum_{n=1}^{\infty} ne^{-nx}$ is well-defined on $(0, +\infty)$ and has derivatives of all orders. Show that f is unbounded on $(0, +\infty)$.

4. Suppose that $A_n, n = 1, 2, \ldots$ are bounded from above non-empty sets of real numbers. Let $a_n = \sup A_n$. Prove that

$$\sup \cup_{n=1}^{\infty} A_n = \sup\{a_n : n = 1, 2, \ldots\}.$$

Note: Make sure to consider the case, where $\cup_{n=1}^{\infty} A_n$ is unbounded from above.

5. Let f_n be sequence of Riemann integrable functions on $[a, b]$. Suppose there exists a Riemann integrable function G on $[a, b]$, such that $|f_n(x)| \leq G(x)$ for all $x \in [a, b]$. Define a sequence of functions

$$F_n(x) = \int_a^x f_n(t)dt$$

Prove that $\{F_n\}$ has an uniformly convergent subsequence.
Hint: Use the Arzela-Ascoli theorem.

6. Let $g(x) : [0, \infty) \to \mathbb{R}$ be a non-negative function such that $\int_0^{\infty} g(x)\, dx = 1$. Show that for any continuous function $f : [0, \infty) \to \mathbb{R}$, the sequence

$$f_n(x) = n \int_0^{\infty} g(ny) f(x + y)\, dy$$

converges uniformly on the compact sets of $[0, +\infty)$ to $f(x)$.

7. Let (X, d) be a metric space, K is a compact subset of X, while L is a closed subset, so that $K \cap L = \emptyset$. Prove that one can separate them by open sets. That is, there exist open sets U, V, so that $K \subset U, L \subset V, U \cap V = \emptyset$.